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Evaluation of Radiances Generated by Solving the Radiative-Transfer Equation with Different Approaches¹

Radiative transfer is the main phenomenon in the basis of several relevant problems of scientific and technological interest. Examples of application of the mathematical and computational modeling of such phenomenon can be found in astronomy, environmental sciences, engineering and medicine among many different areas. The integro-differential equation known as Boltzmann equation describes mathematically the interaction of the radiation with the participating medium, i.e. a medium which may absorb, scatter and emit radiation. Several methods have been developed for the solution of the Boltzmann equation. In the present work we present a comparison of the solutions obtained for the one-dimensional problem with four different methods: (i) Monte Carlo (MC) method; (ii) Discrete Ordinates method (S_N) combined with a finite difference approximation; (iii) Analytical Discrete Ordinates method (AS_N); and (iv) Laplace Transform Discrete Ordinates method (LTS_N). Our final objective is to solve the inverse radiative transfer problem and for that purpose, we want to investigate methods that may provide accurate and fast solutions for the direct problem.

Keywords: radiative transfer, Boltzmann equation, Monte Carlo method, Discrete Ordinates method, Laplace Transform Discrete Ordinates method

Introduction

The formulation and solution of direct and inverse radiative transfer problems are directly related to several relevant applications in a large number of areas of scientific and technological interest such as tomography (Kim and Charette, 2007; Carita Montero et al., 2004), remote sensing and environmental sciences (Spurr et al., 2007; Verhoef and Bach, 2003; Hanan, 2001; Fause et al., 2001), and radiative properties estimation (Sousa et al., 2007; Silva Neto et al., 2007; Zhou et al., 2002), among many others.

Many approaches have been developed for the solution of such problems. Hansen and Travis (1974) and Lenoble (1977) provided excellent reviews on the methods for the solution of the direct radiative transfer problem, and McCormick (1992) did the same with respect to the inverse problem.

In recent years it has been observed a growing interest towards the stochastic Monte Carlo method for the solution of the direct problem (Maurente et al., 2007; Chen and Liou, 2006; Battaglia and Mantovani, 2005; Postlyakov, 2004, 2004a), as well as towards variations of the Discrete Ordinates Method (Çayan and Selçuk, 2007; Chalhoub, 2003, 2005) which was originally proposed by Wick (1943) and Chandrasekhar (1944, 1950). Moreover, some researchers have performed comparisons of different solution strategies in order to identify accurate and fast methods to be used both in the direct and inverse radiative transfer problems (Jensen et al., 2007; Bulgarelli and Doyle, 2004; Chalhoub et al., 2003).

In the present work we present a comparison of the solutions obtained for the direct radiative transfer problem in one-dimensional homogeneous and gray participating media with isotropic scattering using four different methods: (i) a Monte Carlo (MC) method; (ii) the Discrete Ordinates Method combined with a finite difference approximation, here denominated SMDO (Single Mesh Discrete Ordinates); (iii) the Analytical Discrete Ordinates method (AS_N); and (iv) the Laplace Transform Discrete Ordinates Method (LTS_N).

Our main objective is to investigate methods that can provide accurate and fast solutions to the direct problem, in order to be used in the solution of the inverse radiative transfer problem.

Nomenclature

| | |
|----------------------------|---|
| A_k, B_k | = k -th coefficients of the linear system of N_q algebraic equations, dimensionless |
| a_i | = i -th weight of the quadrature order, dimensionless |
| a_j | = j -th weight of the quadrature order, dimensionless |
| a_n | = n -th weight of the quadrature order, dimensionless |
| d^+, d^- | = positive and negative eigenvalues, dimensionless |
| f_1 | = intensity of the isotropic external source of radiation incident at $\tau = 0$ in photons, dimensionless |
| f_2 | = intensity of the isotropic external source of radiation incident at $\tau = \tau_0$ in photons, dimensionless |
| H | = number of photon histories, dimensionless |
| \mathbf{I} | = N -order identity matrix, dimensionless |
| I | = intensity (radiance) of the radiation field in photons, dimensionless |
| I_m | = intensity (radiance) of the radiation field in photons, at the m -th collocation point of the quadrature, dimensionless |
| $I_c^\pm(\tau, \Delta\mu)$ | = photons traveling at a given location τ and within a given polar angle interval $\Delta\mu$, dimensionless |
| $I(\tau, \pm\mu)$ | = average radiance in photons, dimensionless |
| l | = geometrical length, m |
| M | = number of collocation points of the quadrature, dimensionless |
| $\bar{M}_{N_q}^m(s)$ | = LTS_N N_q -order matrix, dimensionless |
| N | = number of nodes for spatial discretization, dimensionless |
| $N_q = M$ | = number of collocation points of the quadrature or quadrature order, dimensionless |
| n | = $N_q/2$, dimensionless |
| $p(l)$ | = probability density function, dimensionless |
| p_1, p_2 | = probabilities, dimensionless |
| R | = probability distribution function, dimensionless |
| R_τ | = random number for the calculation of the distance to collision, dimensionless |

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- R_{ϖ} = random number for the calculation of the single scattering albedo, dimensionless
- R_{μ} = random number for the calculation of the cosine of the scattering angle θ , dimensionless
- R_{φ} = random number for the calculation of the azimuthal angle φ , dimensionless
- S = total source rates, dimensionless
- $x_{0,1,2}, y_{0,1,2}, z_{0,1,2}$ = Cartesian coordinates, dimensionless

Greek Symbols

- θ_0, φ_0 = polar coordinates, dimensionless
- θ = polar angle, dimensionless
- φ = azimuthal angle, dimensionless
- μ = cosine of the polar angle, dimensionless
- μ_j = j -th value of the cosine of the polar angle, dimensionless
- μ_m = m -th value of the cosine of the polar angle, dimensionless
- ϖ = single scattering albedo, dimensionless
- τ = optical variable, dimensionless
- τ_0 = optical thickness, dimensionless
- $\Omega_{0,1,2}^x, \Omega_{0,1,2}^y, \Omega_{0,1,2}^z$ = direction cosines, dimensionless
- σ_a = absorption coefficient, m^{-1}
- σ_s = scattering coefficient, m^{-1}
- σ_t = attenuation coefficient of the medium, m^{-1}
- $\tilde{\beta}(\theta, \varphi)$ = phase function for anisotropic scattering, dimensionless
- $\tilde{\beta}(\theta)$ = phase function for isotropic scattering, dimensionless
- $\Delta\mu$ = polar angle interval, dimensionless
- $\Phi(v_k, \mu_j)$ = j -th component of the eigenvector $\Phi(v_k)$ associated with the eigenvalue $1/v_k$, dimensionless
- $\Phi(-v_k, \mu_j)$ = j -th component of the eigenvector $\Phi(-v_k)$ associated with the eigenvalue $-1/v_k$, dimensionless

The Test Problem

In this work we consider a one-dimensional gray homogeneous, participating medium of optical thickness τ_0 , with transparent boundary surfaces that are subjected to external radiation. It is assumed that the emission of radiation by the medium due to its temperature is negligible in comparison to the intensity of the external incoming radiation. Also the effects of possible differences on the refractive indices of the participating medium and surrounding environment are not taken into account. Our equation of transfer for such problem considering azimuthal symmetry and isotropic scattering within the medium is then given by (Özişik, 1973; Silva Neto and Moura Neto, 2005)

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\varpi}{2} \int_{-1}^1 I(\tau, \mu') d\mu' \tag{1}$$

for $\tau \in (0, \tau_0)$, $\mu \in [-1, 1]$, and $\varpi \in (0, 1)$, subject to the boundary conditions

$$I(0, \mu) = f_1(\mu) \quad \text{and} \quad I(\tau_0, \mu) = f_2(\mu) \tag{2}$$

for $\mu > 0$ and $\mu < 0$, respectively, where $I(\tau, \mu)$ denotes the intensity (radiance) of the radiation field, τ the optical variable, μ the cosine of the polar angle, ϖ the albedo for single scattering, and $f_1(\mu)$ and $f_2(\mu)$ the intensity of the isotropic external sources of radiation incident at $\tau = 0$ and $\tau = \tau_0$, respectively.

A schematic representation of the physical situation considered here is shown in Fig. 1.

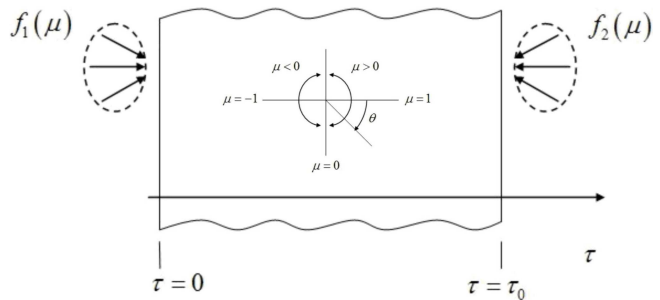


Figure 1. Schematic representation of the 1D participating medium.

In order to solve the direct problem described by Eqs. (1) and (2), we use a Monte Carlo (MC) method and three variations of the Discrete Ordinates Method, proposed by Wick (1943) and Chandrasekhar (1944, 1950): SMDO – Single Mesh Discrete Ordinates; AS_N – Analytical Discrete Ordinates; and LTS_N – Laplace Transform Discrete Ordinates. These four methods, whose corresponding computational codes are referred to as MCPP, SMDO, PEESNA and LTSN, respectively, are described in the following sections.

The MC Method (MCPP Computational Code)

We present a summary of a Monte Carlo method that was based on the works of Cashwell and Everett (1959), and of Carter and Cashwell (1975). In this method, we adopted a physical approach that describes the transfer of radiation by following the history of many individual photons that are generated to represent a light source, until they are absorbed or escape the scattering medium. Quantities describing the photon initial position, the photon trajectories (such as direction of original emission, direction following scattering, and path length between interactions), and quantities describing interaction types (absorption or scattering) may be considered as random variables, each being characterized by some probability density function. In the following paragraphs, we show how to sample each one of the above-mentioned quantities in order to track a photon as it penetrates into the considered medium.

The first required quantities are the position and direction of original emission (point sources), given in terms of the Cartesian coordinates x_0, y_0 and z_0 , and the polar coordinates θ_0 and φ_0 (see Fig. 2), with which we can calculate the first set of direction cosines that are needed to determine the photon position at the first collision.

$$\Omega_0^x = \sin \theta_0 \cos \varphi_0 \tag{3a}$$

$$\Omega_0^y = \sin \theta_0 \sin \varphi_0 \tag{3b}$$

$$\Omega_0^z = \cos \theta_0 \tag{3c}$$

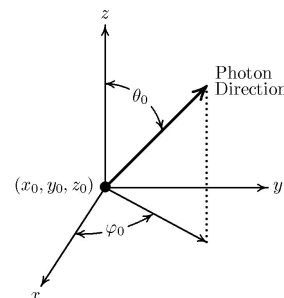


Figure 2. Photon Initial position and direction.

The sampling of the photon paths length, performed by calculating the probability of a collision between the distances l and $l + dl$ along its line of flight, is given by

$$p(l)dl = e^{-\sigma_t l} \sigma_t dl \tag{4}$$

where $p(l)$ denotes the probability density function and σ_t the attenuation coefficient of the medium, which is interpreted as the probability per unit length of a collision. After setting

$$R_\tau = \int_0^l e^{-\sigma_t s} \sigma_t ds = 1 - e^{-\sigma_t l} \tag{5}$$

where R is the probability distribution function and R_τ is a random number, we obtain the expression for the distance to collision as

$$l = -\frac{1}{\sigma_t} \ln(1 - R_\tau) = -\frac{1}{\sigma_t} \ln R_\tau \tag{6}$$

noting that $(1 - R_\tau)$ is distributed in the same manner as R_τ (Carter and Cashwell, 1975). To simplify our calculations, we opted for using the optical length τ instead of the geometrical length l , and so the expression of the distance to collision simply becomes

$$\tau = l\sigma_t = -\ln R_\tau \tag{7}$$

The new position can now be calculated by

$$x_2 = x_1 + \tau\Omega_1^x \tag{8a}$$

$$y_2 = y_1 + \tau\Omega_1^y \tag{8b}$$

$$z_2 = z_1 + \tau\Omega_1^z \tag{8c}$$

where the subscripts 1 and 2 refer to the photon positions at subsequent collisions. Note that for the first collision $x_1 = x_0$, $y_1 = y_0$, $z_1 = z_0$, $\Omega_1^x = \Omega_0^x$, $\Omega_1^y = \Omega_0^y$ and $\Omega_1^z = \Omega_0^z$. With these new positions at hand, we are able to determine whether the particle is still within the system or escaped from it, in which case the sampling process is terminated.

In sampling the interaction types we define the probabilities

$$p_1 = \varpi = \frac{\sigma_s}{\sigma_t} \quad \text{and} \quad p_2 = \frac{\sigma_a}{\sigma_t} \tag{9}$$

with $\sigma_t = \sigma_s + \sigma_a$, where ϖ denotes the single scattering albedo (or the probability of photon survival), σ_s the scattering coefficient and σ_a the absorption coefficient, and by drawing a random number R_ϖ we are able to determine the interaction type. So we let the interaction be an absorption event, considering the particle eliminated from the system, and consequently the sampling process is terminated, when

$$R_\varpi \geq \varpi \tag{10}$$

otherwise the interaction results in scattering.

The sampling of the scattering direction permits the estimation of the scattering angle through the use of the phase function $\tilde{\beta}(\theta, \varphi)$. Here we consider that the phase function is only dependent on the scattering angle θ and that the azimuthal angle φ is uniformly distributed on the interval from 0 to 2π . Thus θ and φ become independent random variables that can be sampled separately. We also consider isotropic scattering, thus $\tilde{\beta}(\theta) = 1/4$ (Mobley, 1994). So, by setting

$$R_\mu = 2\pi \int_{-1}^\mu \tilde{\beta}(\theta) d\mu', \quad (\mu = \cos \theta) \tag{11a}$$

and

$$R_\varphi = \frac{1}{2\pi} \int_0^\varphi d\varphi' \tag{11b}$$

we obtain

$$\cos \theta = 2R_\mu - 1 \tag{12a}$$

and

$$\varphi = 2\pi R_\varphi \tag{12b}$$

The new sets of direction cosines can now be calculated by the equations

$$\Omega_2^x = \sin \theta \cos \varphi \tag{13a}$$

$$\Omega_2^y = \sin \theta \sin \varphi \tag{13b}$$

$$\Omega_2^z = \cos \theta \tag{13c}$$

The above sampling processes are repeated until the photon is absorbed or escapes the system under investigation. Radiometric quantities are computed by a suitable counting of photons through simulated detectors (counters) that are placed on the boundaries and layer interfaces. So by counting photons $I_c^\pm(\tau, \Delta\mu)$ traveling at a given location τ and within a given polar angle interval $\Delta\mu$, we are able to estimate the average radiance

$$I(\tau, \pm\mu) = \frac{S}{H} \frac{I_c^\pm(\tau, \Delta\mu)}{2\pi |\mu| \Delta\mu} \tag{14}$$

where S denotes the total source rates and H the number of photon histories, for $\mu \in [-1, 0)$ and $(0, 1]$, with μ averaged within the interval $\Delta\mu$.

The SMDO Method (SMDO computational code)

This method consists on a combination of the Discrete Ordinates Method with the finite difference method. First, the angular domain is discretized as shown in Fig. 3, and the spatial domain is discretized as shown in Fig. 4.

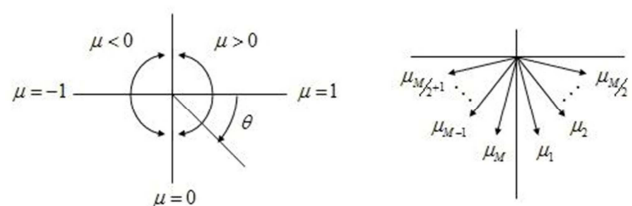


Figure 3. Angular domain discretization.

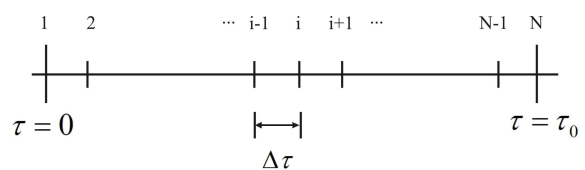


Figure 4. Spatial domain discretization.

The radiation intensities in the angular and spatial discretized domain are represented by

$$I_m^i = I(\tau_i, \mu_m) \quad (15)$$

with $\tau_i = (i - 1)\Delta\tau$, $i = 1, 2, \dots, N$, and $m = 1, 2, \dots, M$, where

$$\Delta\tau = \frac{\tau_0}{N-1}$$

The integral term on the right hand side of Eq. (1) is replaced by a Gauss-Legendre quadrature

$$q^i = \frac{\varpi}{2} \int_{-1}^1 I(\tau, \mu') d\mu' \approx \frac{\varpi}{2} \sum_{n=1}^{N_q} a_n I_n^i \quad (16)$$

where a_n ($n = 1, 2, \dots, N_q$) are the weights of the quadrature. The values of μ_m , $m = 1, 2, \dots, M$ ($M = N_q$), used in the angular domain discretization shown in Fig. 3, are the corresponding collocation points of the quadrature used.

Considering a forward and a backward finite difference discretization of the first term on the left hand side of Eq. (1) given, respectively, by

$$\left. \frac{\partial I(\tau, \mu)}{\partial \tau} \right|_{(\tau_i, \mu_m)} = \frac{I_m^{i+1} - I_m^i}{\Delta\tau} \quad (17a)$$

and

$$\left. \frac{\partial I(\tau, \mu)}{\partial \tau} \right|_{(\tau_i, \mu_m)} = \frac{I_m^i - I_m^{i-1}}{\Delta\tau} \quad (17b)$$

and from Eqs. (1) and (15)-(16) we obtain

$$I_m^{i+1} = \left(1 - \frac{\Delta\tau}{\mu_m}\right) I_m^i + \frac{\Delta\tau}{\mu_m} q^i \quad (18a)$$

and

$$I_m^{i-1} = \left(1 + \frac{\Delta\tau}{\mu_m}\right) I_m^i - \frac{\Delta\tau}{\mu_m} q^i \quad (18b)$$

We performed forward (up to node N) and backward (back to node 1) sweeps, using the discrete boundary conditions expressed as

$$I_m^1 = f_1(\mu_m), \quad m = 1, 2, \dots, \frac{M}{2} \quad (19a)$$

and

$$I_m^N = f_2(\mu_m), \quad m = \frac{M}{2} + 1, \frac{M}{2} + 2, \dots, M \quad (19b)$$

considering the following stopping criterion

$$|(I_m^i)^{k+1} - (I_m^i)^k| < \varepsilon \quad (20)$$

with $i = 1, 2, \dots, N$ and $m = 1, 2, \dots, M$, where ε is a prescribed tolerance and k is the iteration index.

The AS_N Method (PEESNA Computational Code)

In this section, we present a summary of an improved version of the analytical discrete-ordinates method that has been the subject of some recent works (Barichello and Siewert, 1999; Barichello et al., 2000; Chalhoub and Garcia, 2000; Siewert, 2000). In particular, the method incorporates some recently developed techniques for finding particular solutions (Barichello et al., 2000; Siewert, 2000) and dummy-node inclusion (Chalhoub and Garcia, 2000) as its angular interpolation technique. Note that we only present here a simplified version for treating the type of problems described in the test problem section.

For defining our discrete-ordinates version of the problem posed by Eqs. (1) and (2), we begin by introducing a quadrature of order N_q with nodes $\{\mu_j\}$ and weights $\{a_j\}$ to approximate the integral in Eq. (1). The selected quadrature scheme is the double quadrature of order $N_q = 2n$ obtained by applying a standard Gauss-Legendre scheme of order n to each of the half-intervals $[0, 1]$ and $[-1, 0]$. Then we set $\mu = \mu_j$, $j = 1, 2, \dots, N_q$, in the resulting equations to find the discrete-ordinates equations

$$\mu_j \frac{d}{d\tau} I(\tau, \mu_j) + I(\tau, \mu_j) = \frac{\varpi}{2} \sum_{i=1}^{N_q} a_i I(\tau, \mu_i) \quad (21)$$

for $j = 1, 2, \dots, N_q$, and the boundary conditions

$$I(0, \mu_j) = f_1(\mu_j), \quad j = 1, 2, \dots, n \quad (22a)$$

and

$$I(\tau_0, \mu_j) = f_2(\mu_j), \quad j = n+1, n+2, \dots, N_q \quad (22b)$$

Note that the nodes of the quadrature scheme are ordered in such a way that the first n nodes are positive and the remaining n are negative, as shown in Fig. 3.

Making use of the elementary solutions of the discrete-ordinates equations and their orthogonality property developed in Barichello et al. (2000), we can write the general discrete-ordinates solution of order N_q to the problem formulated by Eqs. (21) and (22) as

$$I(\tau, \mu_j) = \sum_{k=1}^n A_k \Phi(v_k, \mu_j) e^{-\frac{\tau}{v_k}} + \sum_{k=n+1}^{N_q} B_k \Phi(-v_k, \mu_j) e^{-(\tau_0 - \tau)/v_k}, \quad (23)$$

for $j = 1, 2, \dots, N_q$. The elementary solutions $\Phi(v_k, \mu_j)$ and $\Phi(-v_k, \mu_j)$ in Eq. (23) are, respectively, the j -th components of the eigenvectors $\Phi(v_k)$ and $\Phi(-v_k)$, associated, respectively, with the eigenvalues $1/v_k$ and $-1/v_k$. Finally, the coefficients $\{A_k\}$ and $\{B_k\}$ are the solutions to the linear system of N_q algebraic equations obtained by imposing that the general solution expressed by Eq. (23) satisfies the boundary conditions expressed by Eqs. (22)

$$\sum_{k=1}^n A_k \Phi(v_k, \mu_j) + \sum_{k=n+1}^{N_q} B_k \Phi(-v_k, \mu_j) e^{-\tau_0/v_k} = I(0, \mu_j) \quad (24a)$$

for $j = 1, 2, \dots, n$, and

$$\sum_{k=1}^n A_k \Phi(v_k, \mu_j) e^{-\tau_0/v_k} + \sum_{k=n+1}^{N_q} B_k \Phi(-v_k, \mu_j) = I(\tau_0, \mu_j) \quad (24b)$$

for $j = n + 1, n + 2, \dots, N_q$. We conclude this summary by pointing out that once the linear system formulated by Eqs. (24) is solved for $\{A_k\}$ and $\{B_k\}$, we can evaluate the radiances with Eq. (23) for any $\tau \in [0, \tau_0]$.

The LTS_N Method (LTSN Computational Code)

The LTS_N scheme appeared in the early nineties in the neutron transport context (Barichello and Vilhena, 1993), and was then extended to radiative transfer problems (Segatto and Vilhena, 1994). Its convergence was established using the C₀-semi group theory (Segatto and Vilhena, 1994). This method applies the Laplace transform on the radiative transfer discrete ordinates equation, Eq. (21). This yields a system of algebraic equations on the Laplace transform parameter s :

$$s\bar{I}^m(s) + \frac{1}{\mu_j} \bar{I}^m(s) = \frac{\varpi}{2\mu_j} \sum_{i=1}^{N_q} a_i \bar{I}^m(s) + I_j^m(0) \quad (25)$$

where $\bar{I}^m(s) \equiv \int_0^{\infty} I^m(\tau) e^{-s\tau} d\tau$. Equation (25) can be formulated in matrix form:

$$\bar{M}_{N_q}^m(s) \bar{I}^m(s) = I^m(0) \quad (26)$$

being $\bar{M}_{N_q}^m(s) = s\mathbf{I} + A^m$, where the N_q -order matrix $\bar{M}_{N_q}^m(s)$ is called the LTS_N matrix, and \mathbf{I} is the N_q -order identity matrix. The entries of the A^m matrix are given by

$$A^m(i, j) = \begin{cases} \frac{1}{\mu_j} - \frac{\varpi a_j}{2\mu_j}, & \text{if } i = j \\ -\frac{\varpi a_i}{2\mu_j}, & \text{if } i \neq j \end{cases} \quad (27)$$

In order to solve the matrix equation (27), we must multiply it by the inverse matrix of $\bar{M}_{N_q}^m(s)$, as follows

$$\bar{I}^m(s) = [\bar{M}_{N_q}^m(s)]^{-1} I^m(0) \Leftrightarrow \bar{I}^m(s) = \bar{B}^m(s) I^m(0) \quad (28)$$

And by applying the Laplace inverse transform yields

$$I^m(\tau) = B^m(\tau) I^m(0) \quad (29)$$

with

$$B^m(\tau) \equiv \mathcal{L}^{-1}[\bar{B}^m(s)] \quad (30)$$

Matrix inversion is usually expensive. The diagonalization method (Segatto et al., 1999) takes advantage of the fact that the LTS_N matrix, Eq. (29), is non-degenerate, i.e. all eigenvalues are distinct, and therefore, A^m can be diagonalized:

$$A^m = X^m D^m (X^m)^{-1} \quad (31)$$

where D^m is a diagonal matrix containing the eigenvalues of A^m , and X^m is the corresponding eigenvectors matrix. Therefore, the matrix B can be expressed as

$$B^m(\tau) = \mathcal{L}^{-1}[(s\mathbf{I} + A^m)^{-1}] = X^m \mathcal{L}^{-1}[(s\mathbf{I} + D^m)^{-1}] (X^m)^{-1} = X^m e^{D^m \tau} (X^m)^{-1} \quad (32)$$

Then, by substituting Eq. (32) into Eq. (29) yields

$$I^m(\tau) = X^m e^{D^m \tau} (X^m)^{-1} I^m(0) = X^m \begin{bmatrix} e^{d^+ \tau} & 0 \\ 0 & e^{d^- \tau} \end{bmatrix}^m (X^m)^{-1} I^m(0) \quad (33)$$

where d^+ and d^- are the positive and negative eigenvalues, respectively. The method, as described by Eqs. (29) and (30), does not work well due to the numerical overflow for large slab thicknesses and/or large values of N_q . This feature can be avoided by a change of variables (Gonçalves et al., 2000). Equation (33) can be written as follows

$$I^m(\tau) = X^m \begin{bmatrix} e^{d^+(\tau-\tau_0)} & 0 \\ 0 & e^{d^-\tau} \end{bmatrix}^m \begin{bmatrix} e^{d^+ \tau_0} & 0 \\ 0 & 1 \end{bmatrix}^m (X^m)^{-1} I^m(0) = B^m(\tau) \xi^m(0) \quad (34)$$

where

$$B^m(\tau) = X^m \begin{bmatrix} e^{d^+(\tau-\tau_0)} & 0 \\ 0 & e^{d^-\tau} \end{bmatrix}^m \quad (35)$$

and

$$\xi^m(0) = \begin{bmatrix} e^{d^+ \tau_0} & 0 \\ 0 & 1 \end{bmatrix}^m (X^m)^{-1} I^m(0) \quad (36)$$

Equation (34) can also be represented by block matrices:

$$\begin{bmatrix} I^1(\tau) \\ I^2(\tau) \end{bmatrix}^m = \begin{bmatrix} B_{11}(\tau) & B_{12}(\tau) \\ B_{21}(\tau) & B_{22}(\tau) \end{bmatrix}^m \begin{bmatrix} \xi^1(0) \\ \xi^2(0) \end{bmatrix}^m \quad (37)$$

with indexes 1 and 2 pointing to either right and left directions of radiances, respectively. This equation can be applied at the position $\tau = \tau_0$, allowing to compute the unknown values $[I^1(\mathbf{0})]^m$ for completing the LTS_N solution.

Numerical Results

Due to the good performance of the AS_N method (PEESNA computational code) in the comparisons of radiances generated by selected methods that were performed in a previous work (Chalhoub et al., 2003), we decided to use its generated results as reference values for the comparisons to be performed in this work.

Five test cases, whose parameters are shown in Table 1, were chosen in order to perform the required comparisons. Besides these parameters we considered a quadrature order $M = N_q = 20$ for the SMDO, PEESNA and LTSN codes. In Table 2, we show the reference values which are the radiances $I(0, \mu)$ and $I(\tau_0, \mu)$ at the selected values of μ , generated by the PEESNA code for the chosen problems.

Table 1. Parameters used to define the Test Case problems.

| Parameter | Meaning | Test Case | | | | |
|-----------|---------------------------------------|-----------|-----|-----|-----|-----|
| | | 1 | 2 | 3 | 4 | 5 |
| ϖ | single scattering albedo | 0.1 | 0.1 | 0.9 | 0.9 | 0.5 |
| τ_0 | optical thickness | 0.5 | 5.0 | 0.5 | 5.0 | 2.0 |
| f_1 | incident radiation at $\tau = 0$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| f_2 | incident radiation at $\tau = \tau_0$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

Table 2. Radiation intensity $I(0, \mu)$ and $I(\tau_0, \mu)$ generated by the PEESNA code (AS_N method) for the chosen Test Cases.

| μ | Test Case 1 | | Test Case 2 | | Test Case 3 | | Test Case 4 | | Test Case 5 | |
|--------|-------------|-----------------|-------------|-----------------|-------------|-----------------|-------------|-----------------|-------------|-----------------|
| | $\tau = 0$ | $\tau = \tau_0$ | $\tau = 0$ | $\tau = \tau_0$ | $\tau = 0$ | $\tau = \tau_0$ | $\tau = 0$ | $\tau = \tau_0$ | $\tau = 0$ | $\tau = \tau_0$ |
| -.9983 | 1.2023E-2 | 0.0 | 1.6460E-2 | 0.0 | 1.6997E-1 | 0.0 | 4.1359E-1 | 0.0 | 1.1329E-1 | 0.0 |
| -.9830 | 1.2317E-2 | 0.0 | 1.6764E-2 | 0.0 | 1.7406E-1 | 0.0 | 4.1812E-1 | 0.0 | 1.1527E-1 | 0.0 |
| -.9426 | 1.2873E-2 | 0.0 | 1.7333E-2 | 0.0 | 1.8180E-1 | 0.0 | 4.2639E-1 | 0.0 | 1.1896E-1 | 0.0 |
| -.8765 | 1.3750E-2 | 0.0 | 1.8208E-2 | 0.0 | 1.9397E-1 | 0.0 | 4.3865E-1 | 0.0 | 1.2457E-1 | 0.0 |
| -.7864 | 1.5047E-2 | 0.0 | 1.9460E-2 | 0.0 | 2.1190E-1 | 0.0 | 4.5529E-1 | 0.0 | 1.3249E-1 | 0.0 |
| -.6750 | 1.6937E-2 | 0.0 | 2.1206E-2 | 0.0 | 2.3786E-1 | 0.0 | 4.7683E-1 | 0.0 | 1.4328E-1 | 0.0 |
| -.5451 | 1.9725E-2 | 0.0 | 2.3639E-2 | 0.0 | 2.7578E-1 | 0.0 | 5.0408E-1 | 0.0 | 1.5784E-1 | 0.0 |
| -.4003 | 2.3970E-2 | 0.0 | 2.7110E-2 | 0.0 | 3.3240E-1 | 0.0 | 5.3823E-1 | 0.0 | 1.7766E-1 | 0.0 |
| -.2446 | 3.0686E-2 | 0.0 | 3.2347E-2 | 0.0 | 4.1782E-1 | 0.0 | 5.8140E-1 | 0.0 | 2.0570E-1 | 0.0 |
| -.0823 | 4.1229E-2 | 0.0 | 4.1435E-2 | 0.0 | 5.2819E-1 | 0.0 | 6.3885E-1 | 0.0 | 2.5007E-1 | 0.0 |
| .0823 | 1.0 | 2.1375E-2 | 1.0 | 6.6726E-5 | 1.0 | 3.1047E-1 | 1.0 | 2.3734E-2 | 1.0 | 2.2502E-2 |
| .2446 | 1.0 | 1.3295E-1 | 1.0 | 8.3119E-5 | 1.0 | 4.3493E-1 | 1.0 | 2.9994E-2 | 1.0 | 2.9163E-2 |
| .4003 | 1.0 | 2.8155E-1 | 1.0 | 1.1019E-4 | 1.0 | 5.4492E-1 | 1.0 | 3.6249E-2 | 1.0 | 4.1542E-2 |
| .5451 | 1.0 | 3.9250E-1 | 1.0 | 2.0825E-4 | 1.0 | 6.2029E-1 | 1.0 | 4.2794E-2 | 1.0 | 6.3509E-2 |
| .6750 | 1.0 | 4.7044E-1 | 1.0 | 6.0521E-4 | 1.0 | 6.7145E-1 | 1.0 | 4.9672E-2 | 1.0 | 9.1293E-2 |
| .7864 | 1.0 | 5.2516E-1 | 1.0 | 1.5370E-3 | 1.0 | 7.0675E-1 | 1.0 | 5.6675E-2 | 1.0 | 1.1958E-1 |
| .8765 | 1.0 | 5.6350E-1 | 1.0 | 2.9783E-3 | 1.0 | 7.3123E-1 | 1.0 | 6.3347E-2 | 1.0 | 1.4481E-1 |
| .9426 | 1.0 | 5.8977E-1 | 1.0 | 4.6397E-3 | 1.0 | 7.4788E-1 | 1.0 | 6.9123E-2 | 1.0 | 1.6504E-1 |
| .9830 | 1.0 | 6.0657E-1 | 1.0 | 6.1245E-3 | 1.0 | 7.5849E-1 | 1.0 | 7.3484E-2 | 1.0 | 1.7938E-1 |
| .9983 | 1.0 | 6.1547E-1 | 1.0 | 7.0790E-3 | 1.0 | 7.6410E-1 | 1.0 | 7.6038E-2 | 1.0 | 1.8743E-1 |

We note that critical parameters in MCPP and SMDO codes had to be adjusted before performing the comparisons. For MCPP the critical parameter is the number of photon histories H and for SMDO it is the number of points in the spatial grid N . The greater the value of these parameters, the more precise the resulting radiances are when compared with the reference values. Table 3 shows for MCPP the number of photon histories H , where, for example, 1K = 10^3 and 1M = 10^6 histories, and for SMDO the grid points N used to reach the established precision (deviations with respect to the reference values lower than 1%). This table only shows the CPU times for the MC method. As for the other methods the CPU times are less than 0.1 second. Note that the codes were executed on a IBM compatible personal computer equipped with a Pentium M 1.7 GHz processor.

Table 3. Critical parameters and CPU times (s).

| Test Case | MCPP | | SMDO |
|-----------|------|---------|------|
| | H | CPU (s) | N |
| 1 | 1M | 1.5 | 50 |
| 2 | 100M | 266.0 | 1000 |
| 3 | 10M | 1.5 | 10 |
| 4 | 1M | 53.0 | 200 |
| 5 | 10M | 27.0 | 200 |

Due to the utilization of test case problems having simple scattering conditions, i.e., isotropic scattering, we were not able to point out the difference in efficiencies presented by the SMDO, PEESNA, and LTSN codes.

To illustrate how the critical parameters were chosen, Fig. 5 shows results obtained by running Test Case 3 with MCPP and SMDO codes using four different values of H and N , respectively. The E values, shown in this figure, represent the global percent deviations that were calculated by a modified version of the Euclidean metric,

$$E = \frac{100}{N_q} \sqrt{\sum_{j=1}^{N_q} \left(\frac{p_j - q_j}{q_j} \right)^2} \tag{38}$$

where $p_j, j = 1, 2, \dots, N_q$, denote the radiances $I(\tau = 0, \mu_j), j = n + 1, n + 2, \dots, N_q$ and $I(\tau = \tau_0, \mu_j), j = 1, 2, \dots, n$, generated with a given critical value and $q_j (j = 1, 2, \dots, N_q)$, those generated with a higher critical value. We also note that for the performed comparisons we chose the critical values that generated results with $E \leq 1\%$.

In Figs. 6-8 we show the radiances generated by the four codes, as well as the E values that represent the global percent deviation in the radiances generated by each one of the codes from the reference values generated by the PEESNA code. The good quality of the approximated solutions obtained with the four methods is observed.

Conclusions

From the comparisons of the radiance generated by MCPP, SMDO, PEESNA, and LTSN codes, we conclude the following:

- As expected, the Monte Carlo method is the most expensive numerical procedure when compared with deterministic techniques.
- The SMDO code requires some analyses to find out the ideal critical parameter, needing a preprocessing scheme.
- The Monte Carlo method requires also a preprocessing in order to determine the lower number of particles that provides good converged solutions.
- AS_N and LTS_N are semi-analytical methods and their solutions are exact for the space variable, as there are no intrinsic truncation errors.
- We plan to solve, in a future work, more realistic problems with anisotropic scattering, represented by complicated scattering functions that contain hundreds of terms and, consequently, requiring extensive CPU times to solve the radiative-transfer equation.

The inverse radiative transfer problems can be formulated as an optimization problem (Silva Neto and Becceneri, 2009; Lobato et al., 2010). Such strategy requires the solution of the direct problem many times until convergence is achieved; therefore, one important feature is to identify methods that provide accurate and fast solutions for the direct problem. The results shown in the present work allow us to say that any one of the used codes: SMDO, PEESNA, and LTSN, is a good choice to fulfill such requirements,

considering an isotropic and homogenous medium, and without a source term. Furthermore, if someone is interested in testing inverse problem solution procedures, one can use the Monte Carlo method output to represent the experimental measurements, minimizing then the inverse crime, in which the same method is used for the simulation of the phenomena of interest in the direct and inverse problems (Kaipio and Somersalo, 2007).

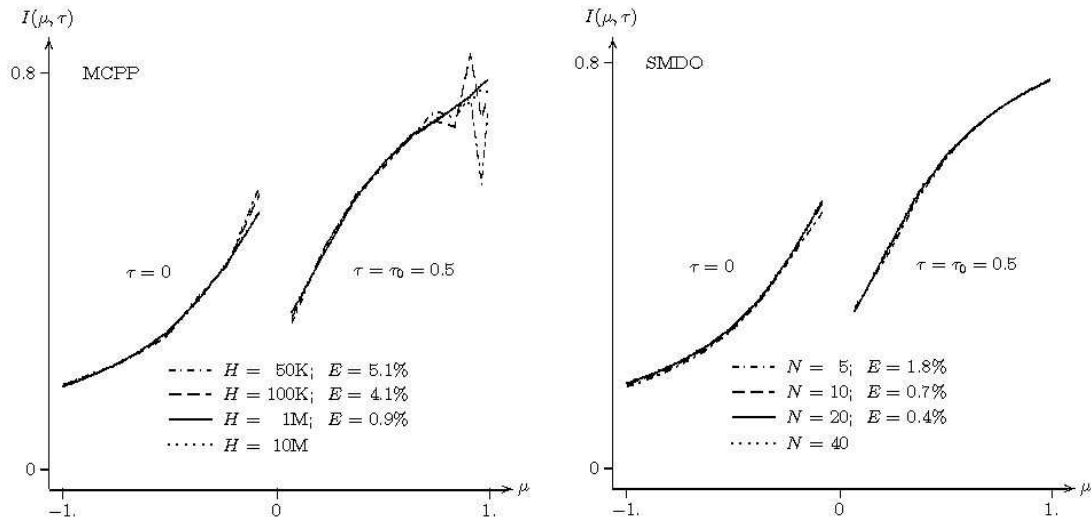


Figure 5. Radiance generated by MCPPP and SMDO with $\omega = 0.9$ and $\tau_0 = 0.5$ (Test Case 3).

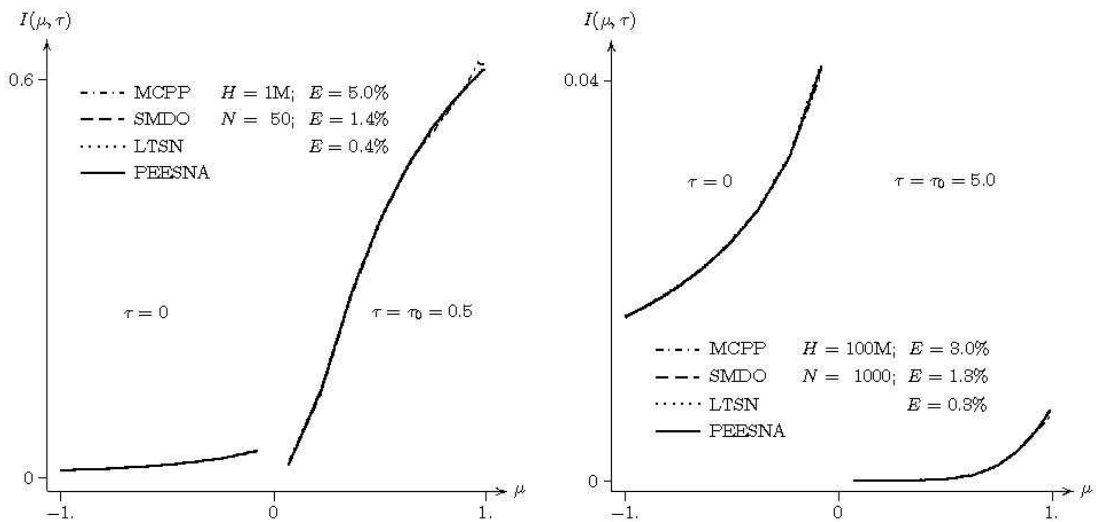


Figure 6. Radiance generated by the four codes with $\omega = 0.1$ and $\tau_0 = 0.5$ (Test Case 1) and 5.0 (Test Case 2).

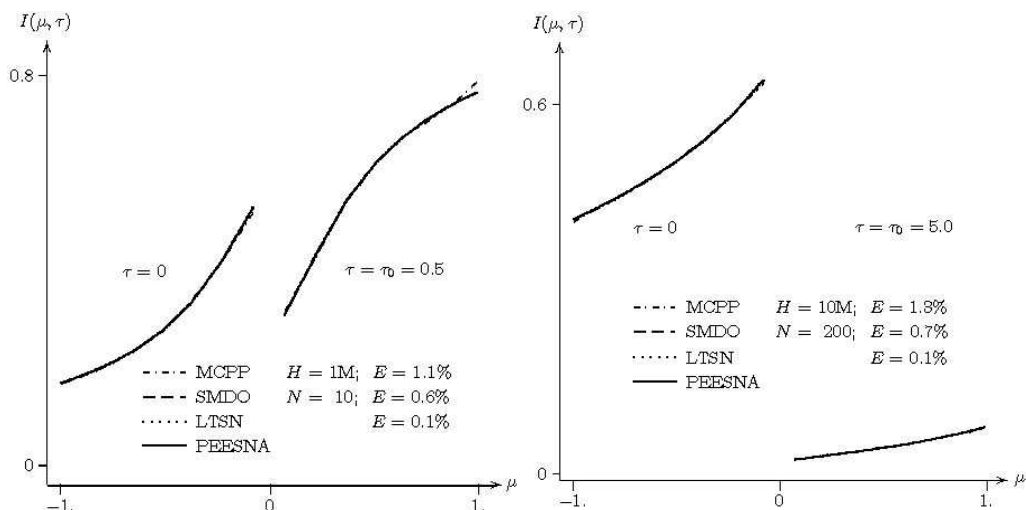


Figure 7. Radiance generated by the four codes with $\omega = 0.9$ and $\tau_0 = 0.5$ (Test Case 3) and 5.0 (Test Case 4).

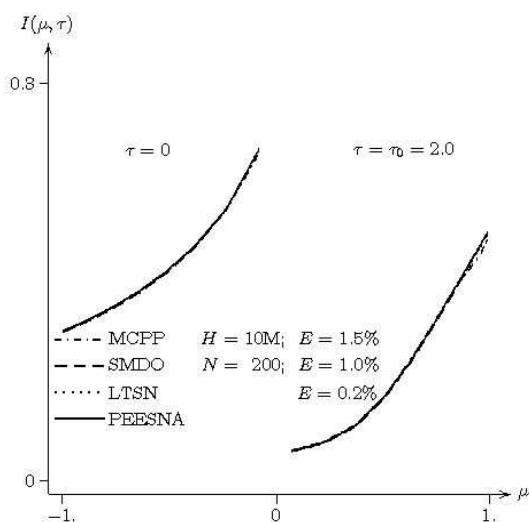


Figure 8. Radiance generated by the four codes with $\omega = 0.5$ and $\tau_0 = 2.0$ (Test Case 5).

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References

Barichello, L.B. and Vilhena, M.T., 1993, "General Approach to One Group One Dimensional Transport Equation", *Kerntechnik*, Vol. 58, No. 3, pp. 182-184.

Barichello, L.B. and Siewert, C.E., 1999, "A Discrete-Ordinate Solution for a Non-Grey Model with Complete Frequency Redistribution", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 62, pp. 665-675.

Barichello, L.B., Garcia, R.D.M. and Siewert, C.E., 2000, "Particular Solutions for the Discrete-Ordinates Method", *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 64, pp. 219-226.

Battaglia, A. and Montovani, S., 2005, "Forward Monte Carlo Computations of Fully Polarized Microwave Radiation in Non-Isotropic Media", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 95, pp. 285-308.

Bulgarelli, B. and Doyle, J.P., 2004, "Comparison Between Numerical Models for Radiative Transfer Simulation in the Atmosphere-Ocean System", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 86, pp. 315-334.

Carita Montero, R.F., Roberty, N.C. and Silva Neto, A.J., 2004, "Reconstruction of a Combination of the Absorption and Scattering Coefficients with a Discrete Ordinates Method Consistent with the Source-Detector System", *Inverse Problems in Engineering*, Vol. 12, No. 1, pp. 81-101.

Carter, L.L. and Cashwell, E.D., 1975, "Particle-Transport Simulation with the Monte Carlo Method", Ed. ERDA Technical Information Center, Oak Ridge, USA.

Cashwell, E.D. and Everett, C.J., 1959, "A Practical Manual of the Monte Carlo Method for Random Walk Problems", Ed. Pergamon, New York, USA.

- Çayan, F.N. and Selçuk, N., 2007, "The Method of Lines Solution of Discrete Ordinates Method for Non-grey Media", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 104, pp. 228-237.
- Chalhoub, E.S. and Garcia, R.D.M., 2000, "The Equivalence between Two Techniques of Angular Interpolation for the Discrete-Ordinates Method", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 64, pp. 517-535.
- Chalhoub, E.S., 2003, "Discrete-Ordinates Solution for Radiative-Transfer Problems", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 76, pp. 193-206.
- Chalhoub, E.S., 2005, "Discrete-Ordinates Solution for Uncoupled Multi-Wavelength Radiative Transfer Problems", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 92, pp. 335-349.
- Chalhoub, E.S., Campos Velho, H.F., Garcia, R.D.M. and Vilhena, M.T., 2003, "A Comparison of Radiances Generated by Selected Methods of Solving the Radiative-Transfer Equation", *Transport Theory and Statistical Physics*, Vol. 32, Nos. 5-7, pp. 473-503.
- Chandrasekhar, S., 1944, "On the Radiative Equilibrium of a Stellar Atmosphere, II", *Astrophys. J.*, Vol. 100, pp. 76-86.
- Chandrasekhar, S., 1950, "Radiative Transfer", Ed. Oxford University Press, London, UK.
- Chen, Y. and Liou, K.N., 2006, "A Monte Carlo Method for 3D Thermal Infrared Radiative Transfer", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 101, pp. 166-178.
- Fause, T., Isaka, H. and Guillemet, B., 2001, "Neural Network Retrieval of Cloud Parameters of Inhomogeneous and Fractional Clouds", *Remote Sensing of Environment*, Vol. 77, pp.123-138.
- Gonçalves, G.A., Segatto, C.F. and Vilhena, M.T., 2000, "The LTSN Particular Solution in a Slab for an Arbitrary Source and Large Order of Quadrature", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 66, No. 3, pp. 271-276.
- Hanan, N.P., 2001, "Enhanced Two-Layer Radiative Transfer Scheme for a Land Surface Model with a Discontinuous Upper Canopy", *Agricultural and Forest Meteorology*, Vol. 109, pp. 265-281.
- Hansen, J.E. and Travis, L.D., 1974, "Light Scattering in Planetary Atmospheres", *Space Science Review*, Vol. 16, pp. 527-610.
- Jensen, K.A., Ripoll, J.-F., Wray, A.A., Joseph, D. and El Hafi, M., 2007, "On Various Modeling Approaches to Radiative Heat Transfer in Pool Fires", *Combustion and Flame*, Vol. 148, pp. 263-279.
- Kaipio, J. and Somersalo, E., 2007, "Statistical Inverse Problems: Discretization, Model Reduction and Inverse Crimes", *Journal of Computational and Applied Mathematics*, Vol. 198, pp. 493-504.
- Kim, H.K. and Charette, A., 2007, "A Sensitivity Function – Based Conjugate Gradient Method for Optical Tomography with the Frequency Domain Equation of Radiative Transfer", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 104, pp. 24-39.
- Lenoble, J. (Ed), 1977, "Standard Procedures to Compute Atmospheric Radiative Transfer in a Scattering Atmosphere", Boulder, CO, National Center for Atmospheric Research.
- Lobato, F.S., Steffen Jr., V. and Silva Neto, A.J., 2010, "A Comparative Study of the Application of Differential Evolution and Simulated Annealing in Inverse Radiative Transfer Problems", *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, Vol. XXXII, No. 5, pp. 518-526.
- Maurente, A., Vielmo, H.A. and França, F.H.R., 2007, "A Monte Carlo Implementation to Solve Radiation Heat Transfer in Non-Uniform Media with Spectrally Dependent Properties", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 108, Issue 2, pp. 295-307.
- McCormick, N.J., 1992, "Inverse Radiative Transfer Problems: A Review", *Nuclear Science and Engineering*, Vol. 112, pp. 185-198.
- Mobley C.D., 1994, "Light and Water – Radiative Transfer in Natural Waters", San Diego:Academic Press.
- Özişik, M.N., 1973, "Radiative Transfer and Interactions with Conduction and Convection", Ed. John Wiley, New York, USA.
- Postlyakov, O.V., 2004, "Linearized Vector Radiative Transfer Model MCC++ for a Spherical Atmosphere", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 88, pp. 297-317.
- Postlyakov, O.V., 2004a, "Radiative Transfer Model MCC++ with Evaluation of Weighting Functions in Spherical Atmosphere for Use in Retrieval Algorithms", *Advances in Space Research*, Vol. 34, pp. 721-726.
- Segatto, C.F. and Vilhena, M.T., 1994, "Extension of the LTSN Formulation for Discrete Ordinates Problem without Azimuthal Symmetry", *Annals of Nuclear Energy*, Vol. 21, No. 11, pp. 701-710.
- Segatto, C.F., Vilhena, M.T., and Gomes, M.G., 1999, "The One-Dimensional LTSn Solution in a Slab with High Degree of Quadrature", *Annals of Nuclear Energy*, Vol. 26, No. 10, pp. 925-934.
- Siewert, C.E., 2000, "A Concise and Accurate Solution to Chandrasekhar's Basic Problem in Radiative Transfer", *Journal of Quantitative Spectroscopy and Radiative Transfer*, Vol. 64, pp. 109-130.
- Silva Neto, A.J. and Becceneri, J.C. (Eds.), 2009, "Nature Inspired Computational Intelligence Techniques", Vol. 41, Notes in Applied Mathematics, SBMAC, São Carlos (In Portuguese).
- Silva Neto, A.J. and Moura Neto, F.D., 2005, "Inverse Problems – Fundamental Concepts and Applications", EdUERJ, Rio de Janeiro. (In Portuguese)
- Silva Neto, A.J., Roberty, N.C., Pinheiro, R.P.F. and Alvarez Acevedo, N.L., 2007, "Inverse Problems Explicit and Implicit Formulations with Applications in Engineering", *Biophysics and Biotechnology, Inverse Problems in Science and Engineering*, Vol. 15, No. 4, pp. 373-411.
- Sousa, F.L., Soeiro, F.J.C.P., Silva Neto, A.J. and Ramos, F.M., 2007, "Application of the Generalized Extremal Optimization Algorithm to an Inverse Radiative Transfer Problem", *Inverse Problems in Science and Engineering*, Vol. 15, No. 7, pp. 699-714.
- Spurr, R., Stamnes, K., Eide, H., Li, W., Zhang, K. and Stamnes, J., 2007, "Simultaneous Retrieval of Aerosols and Ocean Properties: A Classic Inverse Modelling Approach. I – Analytic Jacobians from the Linearized CAO-DISORT Model", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 104, pp. 428-449.
- Verhoef, W. and Bach, H., 2003, "Simulation of Hyperspectral and Directional Radiance Images Using Coupled Biophysical and Atmospheric Radiative Transfer Models", *Remote Sensing of Environment*, Vol. 87, pp. 23-41.
- Wick, G.C., 1943, "Über ebene Diffusionsproblem", *Z. Phys.*, Vol. 120, pp. 702-705.
- Zhou, H.-C., Hou, Y.-B., Chen, D.-L. and Zheng, C.-G., 2002, "An Inverse Radiative Transfer Problem of Simultaneously Estimating Profiles of Temperature and Radiative Parameters from Boundary Intensity and Temperature Measurements", *Journal of Quantitative Spectroscopy & Radiative Transfer*, Vol. 74, pp. 605-620.